DYNAMICS SURVEY OF PHYSICAL PENDULUM WITH TWO DEGREES OF FREEDOM

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Abstract

The paper presents a method of dynamics survey of the physical pendulum which from mechanical point of view is a system with two degrees of freedom. This method is based on matrix computation and it consists in writing the equations of motion for each solid rigid body which compound the system as if it were free. Then these equations of motion will be assembled taking into account the constraint forces. The method may be extended to any systems with three or more degrees of freedom as well. The physical pendulum with two degrees of freedom whose dynamics survey is shown in the paper serves only as a model of the application of the method.

Keywords: dynamics survey, matrix computation, physical pendulum

1. INTRODUCTION

We will consider the physical pendulum with two degrees of freedom which is shown in the figure 1. It is made up of the solid rigid bodies 1 and 2 against which are acting the gravitation forces $\overline{G}_1$ and $\overline{G}_2$ respectively. The solid rigid body "1" is connected with the stand by the joint (hinge) "O_1" and the solid rigid body "2" is connected with the solid rigid body "2" by the joint "O_2".

2. WRITTING THE EQUATIONS OF MOTION

The differential equations which describe the motion of the solid rigid body "i" will be written in the following matrix form:

$$\left[M_{O_i}\right]\cdot\left\{\dot{v}_i\right\} = -\left[\Omega_i\right]\cdot\left\{v_i\right\} + \left\{Q_i\right\} \tag{1}$$

$$\left[M_{O_i}\right] = \left[\begin{array}{c:c} \left[M_i\right] & -\left[S_{O_i}\right] \\ \hdashline \left[S_{O_i}\right] & \left[J_{O_i}\right] \end{array}\right] \tag{2}$$

$$\left[\Omega_i\right] = \left[\begin{array}{c:c} \left[\omega_i\right]\cdot\left[M_i\right] & -\left[\omega_i\right]\cdot\left[S_{O_i}\right] \\ \hdashline \left[S_{O_i}\right]\cdot\left[M_i\right] & \left[\omega_i\right]\cdot\left[J_{O_i}\right] \end{array}\right] \tag{3}$$

$$\left[S_{O_i}\right] = \left[\begin{array}{c:c:c} 0 & 0 & 0 \\ \hdashline 0 & 0 & -m_i\cdot O_iC_i \\ \hdashline 0 & m_i\cdot O_iC_i & 0 \end{array}\right] \tag{4}$$

$$\left[J_{O_i}\right] = \left[\begin{array}{c:c:c} J_{x_i} & 0 & 0 \\ \hdashline 0 & J_{y_i} & 0 \\ \hdashline 0 & 0 & J_{z_i} \end{array}\right] \tag{5}$$

$$\left[\omega_i\right] = \left[\begin{array}{c:c:c} 0 & -\omega_{z_i} & \omega_{y_i} \\ \hdashline \omega_{z_i} & 0 & -\omega_{x_i} \\ \hdashline -\omega_{y_i} & \omega_{x_i} & 0 \end{array}\right] \tag{6}$$

$$\left[M_i\right] = \left[\begin{array}{c:c:c} m_i & 0 & 0 \\ \hdashline 0 & m_i & 0 \\ \hdashline 0 & 0 & m_i \end{array}\right] = m_i\cdot\left[\begin{array}{c:c:c} 1 & 0 & 0 \\ \hdashline 0 & 1 & 0 \\ \hdashline 0 & 0 & 1 \end{array}\right] \tag{7}$$

$$\left\{\dot{v}_i\right\} = \left[\begin{array}{c:c:c:c:c:c} \dot{v}_{O_ix_i} & \dot{v}_{O_iy_i} & \dot{v}_{O_iz_i} & \dot{\omega}_{x_i} & \dot{\omega}_{y_i} & \dot{\omega}_{z_i} \end{array}\right]^T \tag{8}$$

$$\left\{v_i\right\} = \left[\begin{array}{c:c:c:c:c:c} v_{O_ix_i} & v_{O_iy_i} & v_{O_iz_i} & \omega_{x_i} & \omega_{y_i} & \omega_{z_i} \end{array}\right]^T \tag{9}$$

$$\left\{Q_i\right\} = \left[\begin{array}{c:c} \left\{G_i\right\}^T\cdot\left[R_i\right] & \left\{G_i\right\}^T\cdot\left[R_i\right]\cdot\left[r_i\right]^T \end{array}\right]^T \tag{10}$$

$$\left\{G_i\right\} = \left[\begin{array}{c:c:c} m_i\cdot g & 0 & 0 \end{array}\right]^T \tag{11}$$

$$\left[r_i\right] = \left[\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & 0 & -O_iC_i \\ \hline 0 & O_iC_i & 0 \end{array}\right] \quad (12)$$

$$\left[R_i\right] = \left[\begin{array}{c|c|c} \cos(\varphi_i) & -\sin(\varphi_i) & 0 \\ \hline \sin(\varphi_i) & \cos(\varphi_i) & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \quad (13)$$

In the relations from above the index "i" denotes the number of the solid rigid body which compound the system and "m_i" the mass of the solid rigid body "i" from the system.

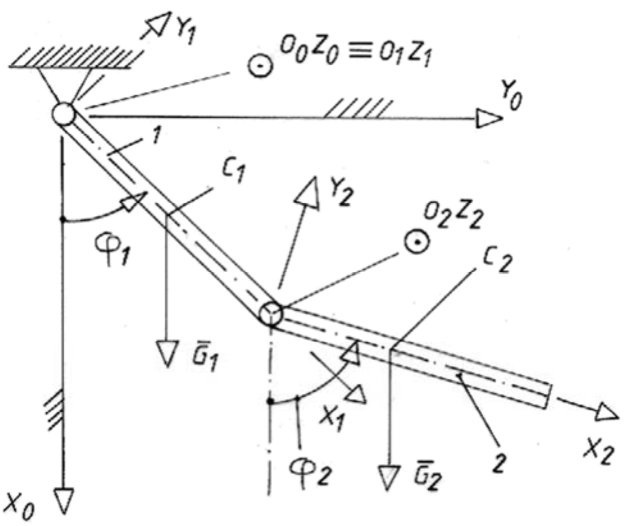


Fig.1 – The figure represents the phisical pendulum with two degrees of freedom

1 – the first solid rigid body of the system
2 – the second solid rigid body of the system

φ_1 - the angle of self-rotation of the first solid rigid body
φ_2 - the angle of self-rotation of the second solid rigid body

3. WRITTING THE CONSTRAINT RELATIONS BETWEEN KINEMATICAL PARAMETERS OF THE FIRST ORDER.

Between kinematical parameters the following matrix constraint relations may be written:

$$[L_\lambda]\cdot\{v\} = \{0\} \quad (14)$$

$$\{v\} = \left[\{v_1\}^T \mid \{v_2\}^T\right]^T \quad (15)$$

$$\{v_1\} = \left[v_{O_1x_1} \mid v_{O_1y_1} \mid v_{O_1z_1} \mid \omega_{x_1} \mid \omega_{y_1} \mid \omega_{z_1}\right]^T \quad (16)$$

$$\{v_2\} = \left[v_{O_2x_2} \mid v_{O_2y_2} \mid v_{O_2z_2} \mid \omega_{x_2} \mid \omega_{y_2} \mid \omega_{z_2}\right]^T \quad (17)$$

$$\left[L_\lambda\right] = \left[\begin{array}{c|c} \left[L_\lambda^{11}\right] & \left[L_\lambda^{12}\right] \\ \hline \left[L_\lambda^{21}\right] & \left[L_\lambda^{22}\right] \end{array}\right] \quad (18)$$

$$\left[L_\lambda^{11}\right] = \left[\begin{array}{c|c} \left[R_1\right] & \overbrace{\left[0\right]}^{3\times3} \\ \hline \underbrace{\left[0\right]}_{2\times3} & \left[\begin{array}{c|c} \left[R_1^{33}\right] & \underbrace{\left\{0\right\}}_{2\times1} \end{array}\right] \end{array}\right] \quad (19)$$

$$\left[L_\lambda^{12}\right] = \underbrace{\left[0\right]}_{5\times6} \tag{20}$$

$$\left[L_\lambda^{21}\right] = \left[\begin{array}{c|c} \left[R_1\right] & \left[R_1\right]\cdot\left[O_1O_2\right]^T \\ \hline \underbrace{\left[0\right]}_{2\times3} & \left[\begin{array}{c|c} \left[R_1^{33}\right] & \underbrace{\{0\}}_{2\times1} \end{array}\right] \end{array}\right] \tag{21}$$

$$\left[L_\lambda^{22}\right] = \left[\begin{array}{c|c} -\left[R_2\right] & \overbrace{\left[0\right]}^{3\times3} \\ \hline \underbrace{\left[0\right]}_{2\times3} & \left[\begin{array}{c|c} \left[R_2^{33}\right] & \underbrace{\{0\}}_{2\times1} \end{array}\right] \end{array}\right] \tag{22}$$

$$\left[R_i^{33}\right] = \left[\begin{array}{c|c} \cos(\varphi_i) & -\sin(\varphi_i) \\ \hline \sin(\varphi_i) & \cos(\varphi_i) \end{array}\right] \tag{23}$$

The index "i" which appears in the above mentioned relations takes values from 1 to 2 (i=1,2).

4. COMPUTATION OF THE CONSTRAINTS MATRIX

The relation (14) may be written under the following equivalent form:

$$\left[L_\lambda^{extended}\right]\cdot\{v\} = [A]\cdot\{\dot{q}\} \tag{24}$$

$$\left[L_\lambda^{extended}\right] = \left[\begin{array}{c|c} \left[L_\lambda^{extended}{}_{11}\right] & \left[L_\lambda^{extended}{}_{12}\right] \\ \hline \left[L_\lambda^{extended}{}_{21}\right] & \left[L_\lambda^{extended}{}_{22}\right] \end{array}\right] \tag{25}$$

$$\left[L_\lambda^{extended}{}_{11}\right] = \left[\begin{array}{c|c} \left[R_1\right] & \overbrace{\left[0\right]}^{3\times3} \\ \hline \underbrace{\left[0\right]}_{3\times3} & \left[R_1\right] \end{array}\right] \tag{26}$$

$$\left[L_\lambda^{extended}{}_{12}\right] = \left[\begin{array}{c|c} \overbrace{\left[0\right]}^{3\times3} & \overbrace{\left[0\right]}^{3\times3} \\ \hline \underbrace{\left[0\right]}_{3\times3} & \underbrace{\left[0\right]}_{3\times3} \end{array}\right] \tag{27}$$

$$\left[L_\lambda^{extended}{}_{21}\right] = \left[\begin{array}{c|c} \left[R_1\right] & \left[R_1\right]\cdot\left[O_1O_2\right]^T \\ \hline \underbrace{\left[0\right]}_{3\times3} & \left[R_1\right] \end{array}\right] \tag{28}$$

$$\left[L_\lambda^{extended}{}_{22}\right] = \left[\begin{array}{c|c} -\left[R_2\right] & \overbrace{\left[0\right]}^{3\times3} \\ \hline \underbrace{\left[0\right]}_{3\times3} & -\left[R_2\right] \end{array}\right] \tag{29}$$

$$[A] = [[A]_1 \mid [A]_2]^T \tag{30}$$

$$[A]_1 = \left[\begin{array}{c|c} \underbrace{[0]}_{2\times5} & [1 \mid 0]^T \end{array}\right] \tag{31}$$

$$[A]_2 = \left[\begin{array}{c|c} \underbrace{[0]}_{2\times5} & [1 \mid -1]^T \end{array}\right] \tag{32}$$

From relation (24) results:

$$\{v\} = \left[L_\lambda^{extended}\right]^{-1}\cdot[A]\cdot\{\dot{q}\} \tag{33}$$

In the relation (33) the following relation will be introduced:

$$[L_\tau] = \left[L_\lambda^{extended}\right]^{-1}\cdot[A] \tag{34}$$

Using the notation (34) the relation (33) becomes:

$$\{v\} = [L_\tau]\cdot\{\dot{q}\} \tag{35}$$

$$\{\dot{q}\} = \left[\omega_{z_1} \mid \omega_{z_2}\right]^T \tag{36}$$

5. WRITTING THE CONSTRAINT RELATIONS BETWEEN KINEMATICAL PARAMETERS OF THE SECOND ORDER.

The relations of constraint between kinematical parameters of the second order will be written under the following matrix form:

$$\left[L_{\lambda}^{extended}\right]\cdot\{\dot{v}\}+\left[\dot{L}_{\lambda}^{extended}\right]\cdot\{v\}=[A]\cdot\{\ddot{q}\} \tag{37}$$

$$\left[\dot{L}_{\lambda}^{extended}{}_{11}\right]=\left[\begin{array}{c|c}[R_1]\cdot[\omega_1] & \overbrace{[0]}^{3\times3}\\ \hline \underbrace{[0]}_{3\times3} & [\omega_1]\cdot[R_1]\end{array}\right] \tag{38}$$

$$\left[\dot{L}_{\lambda}^{extended}{}_{12}\right]=\left[\begin{array}{c|c}\overbrace{[0]}^{3\times3} & \overbrace{[0]}^{3\times3}\\ \hline \underbrace{[0]}_{3\times3} & \underbrace{[0]}_{3\times3}\end{array}\right] \tag{39}$$

$$\left[\dot{L}_{\lambda}^{extended}{}_{21}\right]=\left[\begin{array}{c|c}[R_1][\omega_1] & [R_1][\omega_1][O_1O_2]^T\\ \hline \underbrace{[0]}_{3\times3} & [\omega_1][R_1]\end{array}\right] \tag{40}$$

$$\left[\dot{L}_{\lambda}^{extended}{}_{22}\right]=\left[\begin{array}{c|c}-[R_1][\omega_1] & \overbrace{[0]}^{3\times3}\\ \hline \underbrace{[0]}_{3\times3} & -[\omega_2][R_2]\end{array}\right] \tag{41}$$

The column matrix $\{\dot{v}\}$ which contains the kinematical parameters of the second order will be deduced from relation (37):

$$\{\dot{v}\}=[L_{\tau}]\cdot\{\ddot{q}\}+[\dot{L}_{\tau}]\cdot\{\dot{q}\} \tag{42}$$

In the relation (42) the matrix $[\dot{L}_{\tau}]$ may be expressed as following:

$$[\dot{L}_{\tau}]=-\left[L_{\lambda}^{extended}\right]^{-1}\cdot\left[\dot{L}_{\lambda}^{extended}\right]\cdot[L_{\tau}] \tag{43}$$

In the relation (43) the matrix $[L_{\tau}]$ has the following expression:

$$[L_{\tau}]=\left[L_{\lambda}^{extended}\right]^{-1}\cdot[A] \tag{44}$$

6. WRITTING THE EQUATIONS OF MOTION OF THE MECHANICAL SYSTEM IN THE PRESENCE OF CONSTRAINTS

In the presence of constraints the equations of motion of the mechanical system will have the following matrix form:

$$[M]\cdot\{\dot{v}\}=-[\Omega]\cdot\{v\}+\{Q\}+[L_{\lambda}]^T\cdot\{\lambda\} \tag{45}$$

$$\{\dot{v}\}=\left[\{\dot{v}\}_1^T \;\vdots\; \{\dot{v}\}_2^T\right]^T \tag{46}$$

$$\{v\}=\left[\{v\}_1^T \;\vdots\; \{v\}_2^T\right]^T \tag{47}$$

$$\{\dot{v}\}_1=\left[\dot{v}_{O_1x_1} \;\vdots\; \dot{v}_{O_1y_1} \;\vdots\; \dot{v}_{O_1z_1} \;\vdots\; \dot{\omega}_{x_1} \;\vdots\; \dot{\omega}_{y_1} \;\vdots\; \dot{\omega}_{z_1}\right]^T \tag{48}$$

$$\{\dot{v}\}_2=\left[\dot{v}_{O_2x_2} \;\vdots\; \dot{v}_{O_2y_2} \;\vdots\; \dot{v}_{O_2z_2} \;\vdots\; \dot{\omega}_{x_2} \;\vdots\; \dot{\omega}_{y_2} \;\vdots\; \dot{\omega}_{z_2}\right]^T \tag{49}$$

$$\{v\}_1=\left[v_{O_1x_1} \;\vdots\; v_{O_1y_1} \;\vdots\; v_{O_1z_1} \;\vdots\; \omega_{x_1} \;\vdots\; \omega_{y_1} \;\vdots\; \omega_{z_1}\right]^T \tag{50}$$

$$\{v\}_2=\left[v_{O_2x_2} \;\vdots\; v_{O_2y_2} \;\vdots\; v_{O_2z_2} \;\vdots\; \omega_{x_2} \;\vdots\; \omega_{y_2} \;\vdots\; \omega_{z_2}\right]^T \tag{51}$$

$$[M]=\left[\begin{array}{c|c}[M_{O_1}] & \overbrace{[0]}^{3\times3}\\ \hline \underbrace{[0]}_{3\times3} & [M_{O_2}]\end{array}\right] \tag{52}$$

$$[\Omega]=\left[\begin{array}{c|c}[\Omega_1] & \overbrace{[0]}^{3\times3}\\ \hline \underbrace{[0]}_{3\times3} & [\Omega_2]\end{array}\right] \tag{53}$$

$$\{Q\}=\left[\{Q\}_1^T \;\vdots\; \{Q\}_2^T\right]^T \tag{54}$$

$$\{\lambda\}=[\lambda_1 \;\vdots\; \lambda_2 \;\vdots\; \lambda_3 \;\vdots\; .\;.\;.\;. \;\vdots\; \lambda_8 \;\vdots\; \lambda_9 \;\vdots\; \lambda_{10}]^T \tag{55}$$

In relation (55) the elements of the column matrix $\{\lambda\}$ represents Lagrange's multipliers which are unknown for now. We will multiply the matrix relation (55) to the left with the matrix $[L_{\tau}]^T$ and we will obtain:

$$[L_{\tau}]^T[M]\cdot\{\dot{v}\}=-[L_{\tau}]^T[\Omega]\cdot\{v\}+[L_{\tau}]^T\{Q\} \tag{56}$$

The product between the matrix $[L_\tau]^T$ and $[L_\lambda]^T$ are equal with the null matrix. From this reason this product does not appear any more in relation (56).

$$\underbrace{[L_\tau]^T}_{2\times12}\cdot\underbrace{[L_\lambda]^T}_{12\times10}=\underbrace{[0]}_{2\times10} \quad (57)$$

We will replace the relations (35) and (42) into relation (56) and we will obtain:

$$[\tilde{M}]\cdot\{\ddot{q}\}=[\tilde{\Omega}]\cdot\{\dot{q}\}+\{\tilde{Q}\} \quad (58)$$

In the matrix relation (58) the terms have the followings expressions:

$$[\tilde{M}]=[L_\tau]^T\cdot[M]\cdot[L_\tau] \quad (59)$$

$$[\tilde{\Omega}]=-\left([L_\tau]^T\cdot[\Omega]\cdot[L_\tau]+[L_\tau]^T\cdot[M]\cdot[\dot{L}_\tau]\right) \quad (60)$$

The relation (60) may be written under the following equivalent form:

$$[\tilde{\Omega}]=-[L_\tau]^T\cdot\left([\Omega]\cdot[L_\tau]+[M]\cdot[\dot{L}_\tau]\right) \quad (61)$$

$$\{\tilde{Q}\}=[L_\tau]^T\cdot\{Q\} \quad (62)$$

$$\{\ddot{q}\}=\left[\dot{\omega}_{z_1} \mid \dot{\omega}_{z_2}\right]^T=\{\dot{\omega}\} \quad (63)$$

$$\{\dot{q}\}=\left[\omega_{z_1} \mid \omega_{z_2}\right]^T=\{\omega\} \quad (64)$$

The relation (64) will be replaced into relation (58) and we will obtain:

$$[\tilde{M}]\cdot\{\dot{\omega}\}=[\tilde{\Omega}]\cdot\{\omega\}+\{\tilde{Q}\} \quad (65)$$

Relation (58) represents a system of differential equations of the second order. In order to solve this system it must be transformed into a system of differential equations of the first order.

$$\begin{bmatrix}[\tilde{M}] & \overset{2\times2}{[0]}\\ \underset{2\times2}{[0]} & [I]_2\end{bmatrix}\cdot\begin{Bmatrix}\{\dot{\omega}\}\\ \{\dot{q}\}\end{Bmatrix}=\begin{bmatrix}[\tilde{\Omega}] & [0]\\ [0] & [I]_2\end{bmatrix}\cdot\begin{Bmatrix}\{\omega\}\\ \{\omega\}\end{Bmatrix}+\begin{Bmatrix}\{\tilde{Q}\}\\ \underset{2\times1}{\{0\}}\end{Bmatrix} \quad (66)$$

$[I]_2$ - represents the unit matrix of the second order

The matrix relation (66) represents a system of four differential equations of the first order which can be integrate using numerical methods (Runge-Kutta method for instance). Thus one will find the values of the angular speeds and of the angles of self-rotation respectively:

$$\{\omega\}=\left[\omega_{z_1} \mid \omega_{z_2}\right]^T \quad (67)$$

$$\{q\}=[\varphi_1 \mid \varphi_2]^T \quad (68)$$

ω_{z_1} - represents the angular speed around O_1z_1 axe of the solid rigid body "1"

ω_{z_2} - represents the angular speed around O_2z_2 axe of the solid rigid body "2"

φ_1 - represents the angle of self-rotation of the solid rigid body "1"

φ_2 - represents the angle of self-rotation of the solid rigid body "2"

We will multiply to the left the relation (65) with the matrix $[\tilde{M}]^{-1}$ and we will obtain the values of the angular accelerations:

$$\{\dot{\omega}\}=[\tilde{M}]^{-1}\cdot[\tilde{\Omega}]\cdot\{\omega\}+[\tilde{M}]^{-1}\cdot\{\tilde{Q}\} \quad (69)$$

$$\{\dot{\omega}\}=\left[\dot{\omega}_{z_1} \mid \dot{\omega}_{z_2}\right]^T \quad (70)$$

$\dot{\omega}_{z_1}$ - represents the angular acceleration around O_1z_1 axe of the solid rigid body "1"

$\dot{\omega}_{z_2}$ - represents the angular acceleration around O_2z_2 axe of the solid rigid body "2"

7. DETERMINING THE CONSTRAINT FORCES OF THE MECHANICAL SYSTEM

We will multiply the matrix relation (55) to the left with the matrix $[L_\lambda]^T$ and we will obtain:

$$[L_\lambda][M]\cdot\{\dot{v}\} = -[L_\tau][\Omega]\cdot\{v\} + [L_\lambda]\{Q\} + \\ + [L_\lambda][L_\lambda]^T\{\lambda\} \tag{71}$$

In the relation (71) the followings notations will be introduced:

$$[L_1] = [L_\lambda]\cdot[M] \tag{72}$$

$$[L_2] = [L_\tau]\cdot[\Omega] \tag{73}$$

$$[L_3] = [L_\lambda]\cdot\{Q\} \tag{74}$$

$$[L_4] = [L_\lambda]\cdot[L_\lambda]^T \tag{75}$$

Using the notations given by the relations (72), (73), (74) and (75) the relation (71) will be written thus:

$$[L_1]\cdot\{\dot{v}\} = -[L_2]\cdot\{v\} + [L_3] + [L_4]\cdot\{\lambda\} \tag{76}$$

From the matrix relation (76) we can find the values of Lagrange multipliers:

$$\{\lambda\} = [L_4]^{-1}\cdot([L_1]\{\dot{v}\} + [L_2]\{v\} - [L_3]) \tag{77}$$

$$\{v\} = [L_\tau]\cdot\{\dot{q}\} \tag{78}$$

$$\{\dot{v}\} = [L_\tau]\cdot\{\ddot{q}\} + [\dot{L}_\tau]\cdot\{\dot{q}\} \tag{79}$$

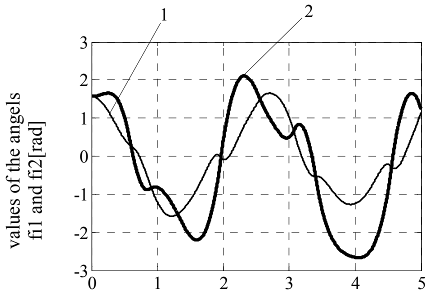
The vectors $\{v\}$ and $\{\dot{v}\}$ which are given by the relations (78) and (79) will be replaced in the relation (77) and we will obtain:

$$\{\lambda\} = [L_4]^{-1}\cdot\begin{pmatrix}[L_1][L_\tau]\{\ddot{q}\} + [L_1][\dot{L}_\tau]\{\dot{q}\} + \\ + [L_2][L_\tau]\{\dot{q}\} - [L_3]\end{pmatrix} \tag{80}$$

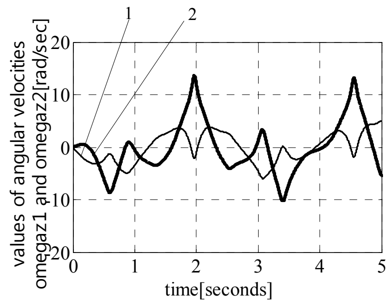
Once Lagrange's multipliers have been determined, the vector of constrained forces will be written as following:

$$\{R^{(const.)}\} = [L_\lambda]^T\cdot\{\lambda\} \tag{81}$$

After the integration of the system we will find out the values of angular speed and angular accelerations. These values are represented in the figure 2 and figure 3.



1-variation of angle fi1
2-variation of angle fi2
fig.2



1-variation of angular speed omegaz1
2-variation of angular speed omegaz2
fig.3

8. REFERENCES

[1] M. Rădoi, E. Deciu, *Mecanica*, Editura Didactică şi Pedagogică Bucureşti, 1981

[2] R. Voinea, D. Voiculescu, V. Ceauşu, *Mecanica*, Editura Didactică şi Pedagogică Bucureşti, 1983

[3] Ştefan Staicu, *Aplications of the matriceal calculus to the mechanics of rigid bodies,* Editura Academiei 1986

[4] Viorel Handra-Luca, Ion Aurel Stoica, *Introducere în teoria mecanismelor*, vol.I şi II Editura Dacia Cluj-Napoca, 1982, 1983

[5] W.Blajer, A Projection Method Approach to Constrained Dynamic Analysis, Journal of Applied Mechanics, Transactions of the ASME, September ,1992 vol.59/pages 643-649

[6] W. Blajer, D.Bestle, W. Schiehlen, An Orthogonal Complement Matrix Formulation for Constrained Multibody Systems, June 1994, vol. 116/pages 423-427 Journal of Mechanical Design